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Chebyshev Rational Approximations to e^{-x} in $[0, +\infty)$ and Applications to Heat-Conduction Problems

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1. INTRODUCTION

For any nonnegative integer m, let π_m denote the collection of all real polynomials of degree at most m, and for any nonnegative integers m and n, let $\pi_{m,n}$ denote the collection of all real rational functions $r_{m,n}(x)$ of the form

$$r_{m,n}(x) \equiv \frac{p_m(x)}{q_n(x)}, \quad \text{where } p_m \in \pi_m \quad \text{and} \quad q_n \in \pi_n. \tag{1.1}$$

With this notation, let

$$\lambda_{m,n} \equiv \inf_{r_{m,n} \in \pi_{m,n}} \{ \sup_{0 \leqslant x < \infty} |r_{m,n}(x) - e^{-x}|\}, \qquad m \leqslant n, \tag{1.2}$$

be the error associated with the best *Chebyshev rational approximation* in $\pi_{m,n}$ to e^{-x} in $[0, +\infty)$. It is known ([*1*], p. 55) that there exists a unique $\hat{r}_{m,n}(x) \in \pi_{m,n}$ such that

$$\lambda_{m,n} = \sup_{0 \leqslant x < \infty} |\hat{r}_{m,n}(x) - e^{-x}|. \tag{1.3}$$

In this paper, we specifically give (in §4) the value of $\lambda_{0,n}$ and $\lambda_{n,n}$ for $0 \leqslant n \leqslant 9$ and $0 \leqslant n \leqslant 14$, respectively, along with the associated minimizing Chebyshev rational approximations $\hat{r}_{n,n}(x)$. These $\lambda_{m,n}$ and $\hat{r}_{n,n}(x)$ were determined because they can be used in the numerical solution of certain heat-conduction problems, and this is illustrated in §3. In a sense, the results of §§3 and 4 continue the original investigation of [7], where only $\lambda_{1,1}$ and $\lambda_{2,2}$ were given.

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Finally, because the computed values of $\lambda_{0,n}$ and $\lambda_{n,n}$ appear to decrease geometrically (cf. Tables I and II of §4), we sought to obtain bounds for the quantities σ_1 and σ_2 defined by

$$\sigma_1 = \overline{\lim_{n\to\infty}} \, (\lambda_{0,n})^{1/n}, \tag{1.4}$$

and

$$\sigma_2 = \overline{\lim_{n\to\infty}} \, (\lambda_{n,n})^{1/n}. \tag{1.5}$$

In §2, we establish by elementary methods the inequalities

$$\tfrac{1}{6} \leqq \sigma_1 \leqq \frac{e^{-\alpha}}{2} = 0.43501 \ldots. \tag{1.6}$$

Here, α stands for the real solution of the equation

$$2\alpha e^{2\alpha+1} = 1, \tag{1.7}$$

and α is approximately

$$\alpha = 0.13923 \ldots.$$

Because $\sigma_2 \leqq \sigma_1$, (1.6) gives also an upper bound for σ_2.

The result (1.6) giving bounds for σ_1, is reminiscent of celebrated results of S. N. Bernstein and J. L. Walsh on Chebyshev approximation of analytic functions by polynomials on finite intervals (cf. [*4*], Theorems 75 and 76), and by rational functions on closed bounded subsets of the complex plane cf. [*8*]). In the latter case, asymptotic results for e^{tx} on $-1 \leqq x \leqq +1$ are given in [*5*]. We do not know, however of any similar results about Chebyshev rational approximation on infinite intervals.

2. Upper Bounds for $\lambda_{m,n}$. Lower Bound for σ_1

1. From the definition in (1.2), it is clear that

$$\lambda_{0,n} \geqq \lambda_{1,n} \geqq \lambda_{2,n} \geqq \ldots \geqq \lambda_{n,n} > 0 \qquad \text{for all } n \geqq 0. \tag{2.1}$$

In particular, for

$$S_n(x) \equiv \sum_{j=0}^{n} \frac{x^j}{j!},$$

the nth partial sum of e^x, it follows that

$$\lambda_{0,n} \leqq \max_{0 \leqq x < \infty} |g_n(x)|, \tag{2.2}$$

where $g_n(x)$ is defined by

$$g_n(x) \equiv \frac{1}{S_n(x)} - e^{-x}. \tag{2.3}$$

We establish

LEMMA 1. *For any integer $n \geqq 0$, we have*

$$0 \leqq g_n(x) \leqq \frac{1}{2^n} \qquad \textit{for } x \geqq 0. \tag{2.4}$$

Proof. Obviously,

$$g_n(0) = 0.$$

Now, let $n \geqq 1$. Since

$$e^x > S_n(x) \qquad \text{for } x > 0,$$

it follows that

$$g_n(x) > 0 \qquad \text{for } x > 0.$$

Let ξ be a positive number at which $g_n(x)$ possesses its maximum in $[0, \infty)$. Then by differentiating (2.3), we have

$$\frac{S_n'(\xi)}{S_n^2(\xi)} = e^{-\xi}.$$

Because of

$$S_n'(x) = S_{n-1}(x),$$

this implies that

$$(g_n(\xi) + e^{-\xi})^2 = e^{-\xi}(g_{n-1}(\xi) + e^{-\xi}).$$

Taking square roots, we derive that

$$\begin{aligned} 0 \leqq g_n(\xi) &= e^{-\xi}\{[1 + g_{n-1}(\xi)\, e^{\xi}]^{1/2} - 1\} \\ &< e^{-\xi} \cdot \tfrac{1}{2} g_{n-1}(\xi)\, e^{\xi} = \tfrac{1}{2} g_{n-1}(\xi). \end{aligned}$$

Therefore,

$$0 \leqq g_n(x) < \tfrac{1}{2} \max_{0 \leqq t < \infty} g_{n-1}(x) \qquad \text{for all } x \leqq 0. \tag{2.5}$$

Thus, as

$$\max_{0 \leqq x < \infty} |g_0(x)| = 1,$$

then (2.4) follows by induction. Q.E.D.

LEMMA 2. *For any integer $n \geqq 0$, we have*

$$\max_{0 \leqq x < \infty} \left| \frac{e^{-\alpha n}}{S_n(x - \alpha n)} - e^{-x} \right| \leqq \frac{1}{(2e^{\alpha})^n}, \tag{2.6}$$

α being the real solution of (1.7).

Proof. We shall prove that

$$|g_n(x)| \leqq \frac{1}{2^n}$$

holds not only for $x \geqq 0$, but even for $x \geqq -\alpha n$. Then, putting

$$x = t - \alpha n,$$

it follows that

$$\left|\frac{1}{S_n(t-\alpha n)} - e^{-(t-\alpha n)}\right| \leqslant \frac{1}{2^n} \qquad \text{for all } t \geqq 0,$$

which gives (2.6).

To prove the above reduced proposition, let $n \geqq 1$. Writing

$$S_n(-y) = e^{-y} - \sum_{j=n+1}^{\infty} (-1)^j \frac{y^j}{j!},$$

we see that for $0 < y \leqslant \alpha n < n+1$, the above series is an alternating series whose terms decrease in modulus monotonically to zero. This gives us that

$$e^{-y} < S_n(-y) < e^{-y} + \frac{y^{n+1}}{(n+1)!}, \qquad n \text{ even}, \tag{2.7}$$

and

$$e^{-y} - \frac{y^{n+1}}{(n+1)!} < S_n(-y) < e^{-y}, \qquad n \text{ odd}, \tag{2.8}$$

for $0 < y \leqslant \alpha n$. To obtain a (positive) lower bound for the left side of (2.8) for $0 < y \leqslant \alpha n$, we observe that

$$e^{-y} - \frac{y^{n+1}}{(n+1)!} \geqslant e^{-\alpha n} - \frac{(\alpha n)^{n+1}}{(n+1)!} = \frac{n^{n+1}}{(n+1)!}\left\{\frac{(n+1)!\, e^{-\alpha n}}{n^{n+1}} - \alpha^{n+1}\right\}$$

$$> \frac{n^{n+1}\, e^{-(1+\alpha)n}}{(n+1)!}\{\sqrt{2\pi n} - \alpha(\alpha\, e^{1+\alpha})^n\},$$

the last inequality following from Stirling's inequality. But since

$$\alpha e^{\alpha+1} = \frac{1}{2e^{\alpha}}$$

from (1.7), and

$$\left\{\sqrt{2\pi n} - \alpha\left(\frac{1}{2e^{\alpha}}\right)^n\right\} > 1$$

for all $n \geqslant 1$, then

$$e^{-y} - \frac{y^{n+1}}{(n+1)!} > \frac{n^{n+1}\, e^{-(1+\alpha)n}}{(n+1)!} \qquad \text{for all } n \geqslant 1, \qquad 0 < y \leqslant \alpha n. \tag{2.9}$$

The inequalities (2.8), (2.9) imply for odd n that

$$0 < \frac{1}{S_n(-y)} - e^y < \frac{1}{e^{-y} - \dfrac{y^{n+1}}{(n+1)!}} - e^y$$

$$= \frac{y^{n+1}\, e^y}{(n+1)!\left(e^{-y} - \dfrac{y^{n+1}}{(n+1)!}\right)}$$

$$\leqq \frac{\alpha^{n+1}\, n^{n+1}\, e^{\alpha n}}{n^{n+1}\, e^{-(1+\alpha)n}} = \alpha(\alpha\, e^{(2\alpha+1)})^n = \frac{\alpha}{2^n}.$$

Consequently,

$$0 < \frac{1}{S_n(-y)} - e^y < \frac{1}{2^n} \qquad \text{for } n \text{ odd}, \qquad 0 < y \leqq \alpha n. \tag{2.10}$$

For n even, one similarly arrives at

$$0 > \frac{1}{S_n(-y)} - e^y > -\frac{y^{n+1}}{(n+1)!}\, e^{2y}$$

$$\geqq -\frac{n^{n+1}\, e^{-n}}{(n+1)!}\, \alpha(\alpha\, e^{(2\alpha+1)})^n$$

$$> -\frac{\alpha}{2^n}.$$

Consequently

$$0 > \frac{1}{S_n(-y)} - e^y > -\frac{1}{2^n} \qquad \text{for } n \text{ even}, \qquad 0 < y \leqslant \alpha n, \tag{2.11}$$

and (2.10) and (2.11) imply the desired inequality (2.6) Q.E.D.

Lemma 2 directly gives us

THEOREM 1. *For any integer $n \geqq 0$, we have*

$$0 < \lambda_{n,n} \leqq \lambda_{n-1,n} \leqq \cdots \leqq \lambda_{0,n} \leqq \frac{1}{(2e^{\alpha})^n}, \tag{2.12}$$

where α is the solution of (1.7).

COROLLARY. *Let $\{m(n)\}_{n=0}^{\infty}$ be any sequence of nonnegative integers such that $0 \leqslant m(n) \leqslant n$ for each $n \geqslant 0$. Then,*

$$\overline{\lim_{n\to\infty}}\, (\lambda_{m(n),n})^{1/n} \leqslant \frac{e^{-\alpha}}{2} = 0.43501\ldots. \tag{2.13}$$

2. Now, we want to show that at least in the case $m = 0$, the speed of convergence of the sequence $\lambda_{m,n}$ is not greater than geometric. Again, we need two lemmas. First, we introduce the quantity

$$K_n = \min_{P_n \in \pi_n} \left\{ \max_{0 \leqslant x \leqslant 2n/3} |P_n(x) - e^x| \right\}. \tag{2.14}$$

LEMMA 3. *Suppose that there exists a sequence of polynomials* $\{Q_n(x)\}_{n=0}^{\infty}$ *with* $Q_n(x) \in \pi_n$ *for all* $n \geqslant 0$, *a real number* $q \geqslant 2$, *and an integer* n_0 *such that*

$$\left| \frac{1}{Q_n(x)} - e^{-x} \right| \leqq \frac{1}{q^n} \qquad \textit{for all } x \geqq 0 \textit{ and for all } n \geqq n_0. \tag{2.15}$$

Then,

$$K_n \leqq \frac{(e^{2/3})^n}{(q\,e^{-2/3})^n - 1} \qquad \textit{for } n \geqq n_0. \tag{2.16}$$

Proof. First, observe that $e^{2/3} < 2$. Then, from (2.15), it follows for $0 \leqq x \leqq \frac{2}{3}n$, $n > n_0$, that

$$0 < e^{-x} - q^{-n} \leqq \frac{1}{Q_n(x)} \leqq e^{-x} + q^{-n},$$

and therefore

$$-\frac{e^x}{e^{-x}q^n + 1} \leqq Q_n(x) - e^x \leqq \frac{e^x}{e^{-x}q^n - 1}.$$

Thus,

$$|Q_n(x) - e^x| \leqq \frac{(e^{2/3})^n}{(q\,e^{-2/3})^n - 1} \qquad \text{for } 0 \leqq x \leqq \tfrac{2}{3}n, \qquad n \leqq n_0,$$

from which (2.16) is evident. Q.E.D.

LEMMA 4. *For any integer* $n \geqq 0$,

$$K_n > \frac{e^{n/3}\, n^{n+1}}{3 \cdot 6^n (n+1)!}. \tag{2.17}$$

Proof. Writing

$$x = \frac{n}{3}(t+1),$$

we see that

$$K_n = \inf_{\tilde{P}_n \in \pi_n} \left\{ \sup_{-1 \leqq t \leqq +1} |\tilde{P}_n(t) - e^{n(t+1)/3}| \right\}.$$

For $t \in [-1, +1]$, we have the representation

$$e^{n(t+1)/3} = e^{n/3}\left(I_0\left(\frac{n}{3}\right) + 2\sum_{\nu=1}^{\infty} I_\nu\left(\frac{n}{3}\right) T_\nu(t)\right).$$

Here, $T_\nu(t)$ denotes the ν-th-Chebyshev polynomial of the first kind and

$$I_\nu(z) \equiv \sum_{\mu=0}^{\infty} \frac{(z/2)^{2\mu+\nu}}{\mu!(\nu+\mu)!}$$

is the Bessel function of order ν with so-called purely imaginary argument. Obviously,

$$I_\nu(x) > 0 \qquad \text{for } x > 0.$$

By a theorem of Hornecker (cf. [*4*], Theorem 66), then

$$K_n \geqq 2\,e^{n/3} \sum_{\mu=0}^{\infty} I_{(2\mu+1)(n+1)}\left(\frac{n}{3}\right).$$

Since

$$I_{n+1}\left(\frac{n}{3}\right) > \frac{n^{n+1}}{6^{n+1}(n+1)!},$$

it follows that

$$K_n > 2\,e^{n/3}\, I_{n+1}\left(\frac{n}{3}\right) > \frac{e^{n/3}\, n^{n+1}}{3\cdot 6^n(n+1)!},$$

which establishes (2.17). Q.E.D.

Now, we are able to prove

THEOREM 2. *For the quantity* σ_1 *defined in* (1.4), *we have*

$$\sigma_1 \geqq \tfrac{1}{6}. \tag{2.18}$$

Proof. By Theorem 1, we know already

$$\sigma_1 < \tfrac{1}{2}.$$

For every number q with

$$\sigma_1 < \frac{1}{q} < \tfrac{1}{2}, \tag{2.19}$$

there exists, by the definition of σ_1, a sequence of polynomials $Q_n(x)$ and an integer n_0 such that the assumptions of Lemma 3 are satisfied. Combining (2.16) and (2.17) we see that for all $n \geqq n_0$, the inequality

$$\frac{e^{n/3}\, n^{n+1}}{3\cdot 6^n(n+1)!} < \frac{e^{2/3n}}{(q\,e^{-2/3})^n - 1}$$

must hold. Using Stirling's formula, i.e.,

$$n! < \sqrt{2\pi n}\, n^n e^{-n} \left(1 + \frac{1}{4n}\right),$$

leads to

$$g^n < (e^{2/3})^n + 3\left[\left(\frac{n+1}{n}\right) 6^n \sqrt{2\pi n}\right]\left(1 + \frac{1}{4n}\right), \qquad n \geqq n_0.$$

Thus, as $e^{2/3} < 2$, it is clear that the above inequality is valid for all $n \geqq n_0$ only if

$$q \leqq 6.$$

Since q is an arbitrary number which has only to satisfy the inequalities (2.19), it is obvious that

$$\sigma_1 \geqq \tfrac{1}{6}. \qquad \text{Q.E.D.}$$

3. Applications to Heat-Conduction Problems

We begin with the matrix differential equation

$$B\frac{d\mathbf{c}(t)}{dt} = -A\mathbf{c}(t) + \mathbf{g}, \qquad t > 0, \tag{3.1}$$

subject to the initial condition

$$\mathbf{c}(0) = \tilde{\mathbf{c}}. \tag{3.2}$$

Here, A and B are assumed to be *commuting Hermitian and positive definite* $N \times N$ matrices, and $\tilde{c} = (\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_N)^T$. The solution $\mathbf{c}(t)$ of (3.1)–(3.2) can be verified to be

$$\mathbf{c}(t) = A^{-1}\mathbf{g} + \exp(-tB^{-1}A)\{\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\}, \qquad \text{for all } t \geqslant 0. \tag{3.3}$$

For any fixed nonnegative integers m and n with $0 \leqslant m \leqslant n$, let $\hat{r}_{m,n}(x) \equiv \hat{p}_{m,n}(x)/\hat{q}_{m,n}(x)$ denote the (m,n)-th Chebyshev rational approximation of e^{-x} in $[0, +\infty)$, i.e.,

$$\sup_{0 \leqslant x < \infty} |\hat{r}_{m,n}(x) - e^{-x}| = \lambda_{m,n}, \tag{3.4}$$

where $\lambda_{m,n}$ is defined in (1.2). Then, we define the (m,n)-th *Chebyshev approximation* $\mathbf{c}_{m,n}(t)$ of $\mathbf{c}(t)$ in (3.3), by

$$\mathbf{c}_{m,n}(t) = A^{-1}\mathbf{g} + \hat{r}_{m,n}(tB^{-1}A)\{\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\}, \quad \text{for all } t \geqslant 0, \tag{3.5}$$

where $\hat{r}_{m,n}(tB^{-1}A)$ is the matrix formally given by

$$(\hat{q}_{m,n}(tB^{-1}A))^{-1} \cdot (\hat{p}_{m,n}(tB^{-1}A)).$$

From (3.3) and (3.5), we have

$$\mathbf{c}_{m,n}(t) - \mathbf{c}(t) = (\hat{r}_{m,n}(tB^{-1}A) - \exp(-tB^{-1}A))\{\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\}, \quad t \geqslant 0. \quad (3.6)$$

We now associate with the positive definite Hermitian matrix B of (3.1), the particular vector norm

$$\|\mathbf{c}\|_B^2 \equiv \mathbf{c}^* B \mathbf{c} = \|B^{1/2}\mathbf{c}\|_2^2, \quad \text{where } \|\mathbf{v}\|_2^2 \equiv \mathbf{v}^* \cdot \mathbf{v}. \quad (3.7)$$

For any $N \times N$ matrix D, the induced operator norm of D is then

$$\|D\|_B \equiv \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|D\mathbf{x}\|_B}{\|\mathbf{x}\|_B} = \|B^{1/2} DB^{-1/2}\|_2 \equiv \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|B^{1/2} DB^{-1/2}\mathbf{x}\|_2}{\|\mathbf{x}\|_2}.$$

Using the facts that A and B are commuting Hermitian matrices, and the polynomials $\hat{p}_{m,n}(x)$ and $\hat{q}_{m,n}(x)$ are both real, we can write

$$\begin{aligned} \|\hat{r}_{m,n}(tB^{-1}A) - \exp(-tB^{-1}A)\|_B &= \|\hat{r}_{m,n}(tB^{-1}A) - \exp(-tB^{-1}A)\|_2 \\ &= \max_{1 \leqslant i \leqslant N} |\hat{r}_{m,n}(t\lambda_i) - e^{-t\lambda_i}|, \quad \text{for all } t \geqslant 0, \end{aligned}$$

where $\{\lambda_i\}_{i=1}^N$ denote the positive eigenvalues of $B^{-1}A$. But as $t\lambda_i \in [0, +\infty)$ for any nonnegative t and any eigenvalue λ_i, we evidently have from (3.4) that

$$\|\hat{r}_{m,n}(tB^{-1}A) - \exp(-tB^{-1}A)\|_B \leqslant \lambda_{m,n}, \quad \text{for all } t \geqslant 0. \quad (3.8)$$

Thus, taking norms in (3.6), gives us the *global* error bound

$$\|\mathbf{c}_{m,n}(t) - \mathbf{c}(t)\|_B \leqslant \lambda_{m,n}\|\tilde{\mathbf{c}} - A^{-1}\mathbf{g}\|_B, \quad \text{for all } t \geqslant 0. \quad (3.9)$$

To indicate how the inequality (3.9) can be used in the numerical solution of parabolic partial differential equations, we consider here the solution $u(x, t)$ of the simple one-dimensional heat-conduction problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + r(x), \quad 0 < x < 1, \quad t > 0, \quad (3.10)$$

subject to the boundary conditions

$$u(0, t) = u(1, t) = 0, \quad \text{for all } t > 0, \quad (3.11)$$

and the initial condition

$$u(x, 0) = \tilde{u}(x), \quad 0 \leqslant x \leqslant 1, \quad (3.12)$$

where $r(x)$ and $\tilde{u}(x)$ are given real functions on $[0, 1]$. We remark that similar applications are valid in higher dimensions.

For any fixed positive integer N, let $h = 1/(N+1)$, and let $\{w_i(x)\}_{i=1}^N$ be the piecewise-linear functions defined by

$$w_i(x) = \left\{ \begin{array}{ll} 1 - \left(\dfrac{x - ih}{h}\right), & ih \leqslant x \leqslant (i+1)h, \\ 1 + \left(\dfrac{x - ih}{h}\right), & (i-1)h \leqslant x \leqslant ih, \\ 0, & x \notin [(i-1)h, (i+1)h] \end{array} \right\}, \qquad 1 \leqslant i \leqslant N. \tag{3.13}$$

The set S of all real linear combinations of the $w_i(x)$'s is known in the literature as an *Hermite space* (cf. [2], §6). All functions of S vanish at the endpoints of $[0,1]$.

The *semi-discrete Galerkin approximation* (cf. [6])

$$\hat{w}(x,t) \equiv \sum_{i=1}^{N} c_i(t)\, w_i(x), \qquad 0 \leqslant x \leqslant 1, \qquad t \geqslant 0, \tag{3.14}$$

of the solution $u(x,t)$ of (3.10)–(3.12), is determined by solving the matrix differential equation (3.1)–(3.2) for the functions $c_i(t)$, $1 \leqslant i \leqslant N$, where the matrices $B = (b_{i,j})$ and $A = (a_{i,j})$ have their entries explicitly defined by

$$b_{i,j} = \int_0^1 w_i(x)\, w_j(x)\, dx; \qquad a_{i,j} = \int_0^1 w_i'(x)\, w_j'(x)\, dx, \qquad 1 \leqslant i,j \leqslant N, \tag{3.15}$$

and where the vector $\mathbf{g}$ of (3.1) has components g_i defined by

$$g_i = \int_0^1 r(x)\, w_i(x)\, dx, \qquad 1 \leqslant i \leqslant N. \tag{3.16}$$

The vector $\tilde{\mathbf{c}}$ of (3.2) is determined from the coefficients of the best L^2-approximation in S of $\tilde{u}(x)$ of (3.12), i.e.,

$$\inf_{s \in S} \|\tilde{u} - s\|_{L^2[0,1]} = \left\| \tilde{u}(x) - \sum_{i=1}^{N} \tilde{c}_i\, w_i(x) \right\|_{L^2[0,1]}. \tag{3.17}$$

From (3.15), it can be verified that A and B are commuting real tridiagonal symmetric positive definite matrices, so that the inequality of (3.9) is applicable.

Based on energy-type inequalities, it can be deduced from [6], Theorem 1, that for $r(x)$ of (3.10) and $\tilde{u}(x)$ of (3.12) sufficiently smooth, there exists a constant K, independent of h and t, such that

$$\|\hat{w}(\cdot,t) - u(\cdot,t)\|_{L^2[0,1]} \leqslant Kh^2, \qquad \text{for all } t \geqslant 0. \tag{3.18}$$

On the other hand, for any $0 \leqslant m \leqslant n$, define the (m,n)th *Chebyshev-Galerkin approximation* of the solution of (3.10)–(3.12), as

$$\hat{w}_{m,n}(x,t) \equiv \sum_{i=1}^{N} c_{m,n,i}(t)\, w_i(x), \tag{3.19}$$

where the functions $c_{m,n,i}(t)$ are the components of the vector $\mathbf{c}_{m,n}(t)$ of (3.5). Now, using the definitions of (3.7) and (3.15), we verify that

$$\|\hat{w}_{m,n}(\cdot,t)-\hat{w}(\cdot,t)\|^2_{L^2[0,1]}=\int_0^1\left\{\sum_{i=1}^{N}(c_{m,n,i}(t)-c_i(t))\,w_i(x)\right\}^2 dx$$
$$=\|\mathbf{c}_{m,n}(t)-\mathbf{c}(t)\|_B{}^2. \tag{3.20}$$

Hence, from (3.9), we have

$$\|\hat{w}_{m,n}(\cdot,t)-\hat{w}(\cdot,t)\|_{L^2[0,1]}\leqslant\lambda_{m,n}\|\tilde{\mathbf{c}}-A^{-1}\mathbf{g}\|_B,\qquad\text{for all } t\geqslant 0. \tag{3.21}$$

Thus, combining (3.18) and (3.21) gives

$$\|\hat{w}_{m,n}(\cdot,t)-u(\cdot,t)\|_{L^2[0,1]}\leqslant Kh^2+\lambda_{m,n}\|\tilde{\mathbf{c}}-A^{-1}\mathbf{g}\|_B,\qquad\text{for all } t\geqslant 0. \tag{3.22}$$

The point of this global error analysis is that $\hat{w}_{m,n}(x,t)$ can be calculated for *any* $t\geqslant 0$ in just one step, in contrast with standard difference methods which arrive at an approximation for $u(x,m\Delta t)$ only after all intermediate approximations of $u(x,j\Delta t)$, $1\leqslant j\leqslant m$, are computed.

We also remark that the difficult part in determining $\mathbf{c}_{m,n}(t)$ of (3.5) consists of solving the linear system of equations:

$$\hat{q}_{m,n}(tB^{-1}A)(\mathbf{c}_{m,n}(t)-A^{-1}\mathbf{g})=\hat{p}_{m,n}(tB^{-1}A)(\tilde{\mathbf{c}}-A^{-1}\mathbf{q}). \tag{3.23}$$

Since $\hat{p}_{m,n}(tB^{-1}A)$ enters into the computation of $\mathbf{c}_{m,n}(t)$ only as a matrix factor, there is little to be gained computationally by choosing $m<n$ in (3.5). For this basic reason, we were initially interested in the values of $\lambda_{n,n}$, as in [7].

4. The Constants $\lambda_{n,n}$ and $\lambda_{0,n}$

In this section, we give the explicit values of $\lambda_{0,n}$, $0\leqslant n\leqslant 9$, in Table I, and of $\lambda_{n,n}$, $0\leqslant n\leqslant 14$, in Table II. These numbers (and the associated rational functions $\hat{r}_{n,n}(x)$) were determined by using a Remez-type algorithm ([9], p. 173). The actual algorithm used is fully described in Cody, Fraser, and Hart [3].

Table I

n	$\lambda_{0,n}$
0	5.000 (−01)
1	9.357 (−02)
2	2.307 (−02)
3	6.353 (−03)
4	1.848 (−03)
5	5.553 (−04)
6	1.703 (−04)
7	5.294 (−05)
8	1.663 (−05)
9	5.264 (−06)

TABLE II

n	$\lambda_{n,n}$
0	5.000 (−01)
1	6.685 (−02)
2	7.359 (−03)
3	7.994 (−04)
4	8.653 (−05)
5	9.346 (−06)
6	1.008 (−06)
7	1.087 (−07)
8	1.172 (−08)
9	1.263 (−09)
10	1.361 (−10)
11	1.466 (−11)
12	1.579 (−12)
13	1.701 (−13)
14	1.832 (−14)

The following functions $r_{n,n}(x)$, $0 \leqslant n \leqslant 14$, constitute a partial *Walsh Table* (cf. [*4*], p. 162) for Chebyshev rational approximations of e^{-x} in $[0, +\infty)$.

TABLE III

$$e^{-x} \simeq \sum_{i=0}^{n} p_i x^i \Big/ \sum_{i=0}^{n} q_i x^i, \qquad 0 \leqslant x < \infty$$

i	p_i		q_i	
		$n = 1$		
0	1.0669	(00)	1.0000	(00)
1	−1.1535	(−01)	1.7275	(00)
		$n = 2$		
0	9.92641	(−01)	1.00000	(00)
1	−1.88332	(−01)	6.69295	(−01)
2	4.21096	(−03)	5.72258	(−01)
		$n = 3$		
0	1.00079 9	(00)	1.00000 0	(00)
1	−2.23657 8	(−01)	7.98292 3	(−01)
2	1.24996 2	(−02)	2.20411 5	(−01)
3	−9.98100 9	(−05)	1.24858 0	(−01)

TABLE III—*continued*

i	p_i		q_i	
		$n = 4$		
0	9.99913 47	(−01)	1.00000 00	(00)
1	−2.40253 73	(−01)	7.56683 22	(−01)
2	1.84005 09	(−02)	2.91754 68	(−01)
3	−4.49812 30	(−04)	4.57502 12	(−02)
4	1.67651 42	(−06)	1.93769 80	(−02)
		$n = 5$		
0	1.00000 935	(00)	1.00000 000	(00)
1	−2.50230 902	(−01)	7.50174 555	(−01)
2	2.24805 919	(−02)	2.69910 157	(−01)
3	−8.33629 264	(−04)	6.76687 392	(−02)
4	1.07797 622	(−05)	6.93457 968	(−03)
5	−2.19125 327	(−08)	2.34468 866	(−03)
		$n = 6$		
0	9.99998 991	(−01)	1.00000 000	(00)
1	−2.56774 988	(−01)	7.43173 208	(−01)
2	2.53896 499	(−02)	2.68982 436	(−01)
3	−1.17690 441	(−03)	6.15930 326	(−02)
4	2.48209 105	(−05)	1.13649 362	(−02)
5	−1.90699 255	(−07)	8.25674 222	(−04)
6	2.34264 503	(−10)	2.32303 566	(−04)
		$n = 7$		
0	1.00000 0109	(00)	1.00000 0000	(00)
1	−2.61399 8104	(−01)	7.38606 6424	(−01)
2	2.75489 3180	(−02)	2.66094 7238	(−01)
3	−1.46758 9943	(−03)	6.22100 6831	(−02)
4	4.06054 4787	(−05)	1.02296 0372	(−02)
5	−5.37067 6308	(−07)	1.48784 8134	(−03)
6	2.65391 0891	(−09)	8.08876 9796	(−05)
7	−2.11893 3743	(−12)	1.94833 4848	(−05)

TABLE III—*continued*

i	p_i		q_i	
		$n = 8$		
0	9.99999 98828	(−01)	1.00000 00000	(00)
1	−2.64834 06521	(−01)	7.35165 14516	(−01)
2	2.92069 90785	(−02)	2.64380 63240	(−01)
3	−1.71076 69530	(−03)	6.17188 04886	(−02)
4	5.63076 21623	(−05)	1.05208 36925	(−02)
5	−1.01477 31374	(−06)	1.32834 62347	(−03)
6	9.00129 46140	(−09)	1.59103 92054	(−04)
7	−3.03122 44065	(−11)	6.72702 00039	(−06)
8	1.66078 92788	(−14)	1.41677 26615	(−06)
		$n = 9$		
0	1.00000 00012 6	(00)	1.00000 00000 0	(00)
1	−2.67485 66991 9	(−01)	7.32514 42527 7	(−01)
2	3.05175 28366 6	(−02)	2.63030 80259 5	(−01)
3	−1.91477 63922 5	(−03)	6.15308 52837 4	(−02)
4	7.11036 34252 9	(−05)	1.04926 24819 4	(−02)
5	−1.56780 17352 5	(−06)	1.39500 66714 0	(−03)
6	1.95356 66646 4	(−08)	1.41160 60689 2	(−04)
7	−1.22095 56914 1	(−10)	1.43514 07391 1	(−05)
8	2.92870 66373 4	(−13)	4.85965 89227 3	(−07)
9	−1.14850 40902 2	(−16)	9.09160 46659 0	(−08)
		$n = 10$		
0	9.99999 99986 39	(−01)	1.00000 00000 00	(00)
1	−2.69593 55382 19	(−01)	7.30406 43483 33	(−01)
2	3.15778 64047 17	(−02)	2.61984 45116 28	(−01)
3	−2.08723 02875 56	(−03)	6.13598 78524 81	(−02)
4	8.46946 26115 79	(−05)	1.05222 25667 19	(−02)
5	−2.15295 78934 24	(−06)	1.40044 18324 30	(−03)
6	3.35954 01052 85	(−08)	1.51629 42863 89	(−04)
7	−3.02437 91657 93	(−10)	1.26706 04218 95	(−05)
8	1.38351 22001 13	(−12)	1.11798 57266 64	(−06)
9	−2.44794 47827 24	(−15)	3.10240 81250 45	(−08)
10	7.10595 74433 07	(−19)	5.22077 71857 74	(−09)

TABLE III—*continued*

i	p_i		q_i	
		$n = 11$		
0	1.00000 00000 147	(00)	1.00000 00000 000	(00)
1	−2.71308 69737 149	(−01)	7.28691 30396 820	(−01)
2	3.24525 83980 923	(−02)	2.61143 86818 246	(−01)
3	−2.23434 38385 867	(−03)	6.12306 52273 991	(−02)
4	9.70327 53192 328	(−05)	1.05372 96909 746	(−02)
5	−2.74176 69166 461	(−06)	1.41501 45103 337	(−03)
6	5.02362 65041 453	(−08)	1.53331 52911 587	(−04)
7	−5.77549 91658 630	(−10)	1.39348 62063 194	(−05)
8	3.88619 42441 125	(−12)	9.83157 36139 162	(−07)
9	−1.34133 12302 919	(−14)	7.65774 19206 722	(−08)
10	1.80098 07948 555	(−17)	1.77400 34787 683	(−09)
11	−3.97762 94455 404	(−21)	2.71280 51110 139	(−10)
		$n = 12$		
0	9.99999 99999 8420	(−01)	1.00000 00000 0000	(00)
1	−2.72732 01038 1007	(−01)	7.27267 98946 2094	(−01)
2	3.31862 74887 8945	(−02)	2.60454 26687 4069	(−01)
3	−2.36102 86093 3434	(−03)	6.11258 94397 4162	(−02)
4	1.08182 04721 4783	(−04)	1.05516 59564 1775	(−02)
5	−3.31706 70455 2847	(−06)	1.42483 77048 4151	(−03)
6	6.85640 06647 2736	(−08)	1.56277 80799 5398	(−04)
7	−9.40255 67465 0549	(−10)	1.41984 90819 4392	(−05)
8	8.21592 17852 2494	(−12)	1.10762 17782 0598	(−06)
9	−4.24605 37294 1828	(−14)	6.71223 49299 4341	(−08)
10	1.13357 45322 5507	(−16)	4.67724 66320 0848	(−09)
11	−1.18241 93272 9819	(−19)	9.18591 91007 0282	(−11)
12	2.03287 74252 3846	(−23)	1.28708 77757 5584	(−11)
		$n = 13$		
0	1.00000 00000 00170	(00)	1.00000 00000 00000	(00)
1	−2.73931 40321 02750	(−01)	7.26068 59680 79845	(−01)
2	3.38101 16410 46875	(−02)	2.59878 71290 03795	(−01)
3	−2.47106 93187 70823	(−03)	6.10400 14004 83738	(−02)
4	1.18246 13397 91637	(−04)	1.05636 62149 46976	(−02)
5	−3.86917 56932 66464	(−06)	1.43339 64981 47585	(−03)
6	8.78467 34854 71303	(−08)	1.58368 07943 13270	(−04)
7	−1.37667 89576 47893	(−09)	1.46094 52103 19625	(−05)
8	1.45460 93630 79049	(−11)	1.13726 98219 71446	(−06)
9	−9.90411 24433 78351	(−14)	7.74828 74238 93217	(−08)
10	4.02014 83515 52472	(−16)	4.08864 75641 62874	(−09)
11	−8.47538 24867 61699	(−19)	2.57666 92606 40894	(−10)
12	7.00505 36680 34527	(−22)	4.34551 90724 16265	(−12)
13	−9.55806 72950 74149	(−26)	5.61852 05844 48164	(−13)

TABLE III—*continued*

i	p_i		q_i	
		$n = 14$		
0	9.99999 99999 99816 8	(−01)	1.00000 00000 00000 0	(00)
1	−2.74956 04296 30004 3	(−01)	7.25043 95703 48866 6	(−01)
2	3.43469 84175 67147 5	(−02)	2.59390 94125 01801 2	(−01)
3	−2.56744 39819 02861 8	(−03)	6.09681 85127 28359 5	(−02)
4	1.27340 70715 23318 1	(−04)	1.05740 49161 69115 6	(−02)
5	−4.39328 08492 51123 6	(−06)	1.44055 27154 58731 6	(−03)
6	1.07532 02054 48522 7	(−07)	1.60192 11440 61219 0	(−04)
7	−1.87102 55961 08945 3	(−09)	1.49082 34724 80024 2	(−05)
8	2.28495 15765 30015 5	(−11)	1.18202 91576 35577 2	(−06)
9	−1.90386 40942 83534 5	(−13)	8.01639 97982 35750 3	(−08)
10	1.03101 51365 35049 5	(−15)	4.83618 00878 64828 1	(−09)
11	−3.34902 80333 66753 3	(−18)	2.24720 30400 42852 9	(−10)
12	5.67438 25539 52350 1	(−21)	1.29228 02705 79277 9	(−11)
13	−3.77945 23874 50329 5	(−24)	1.89218 38854 02244 9	(−13)
14	4.16098 26642 37661 3	(−28)	2.27106 80218 89129 5	(−14)

REFERENCES

1. N. I. ACHIESER, "Theory of Approximation." Frederick Ungar Publishing Co., New York, 1956. Translated by C. J. Hyman.

2. P. G. CIARLET, M. H. SCHULTZ, AND R. S. VARGA, Numerical methods of high-order accuracy for nonlinear boundary value problems. I. One dimensional problem. *Numer. Math.* **9** (1967), 394–430.

3. W. J. CODY, W. FRASER, AND J. F. HART, Rational Chebyshev approximation using linear equations. *Numer. Math.* To appear.

4. GÜNTER MEINARDUS, "Approximation of Functions: Theory and Numerical Methods." Springer-Verlag, New York, 1967. Translated by L. L. Schumaker.

5. GÜNTER MEINARDUS, Abschätzungen der Minimalabweichung bei rationaler Approximation. *In* "Funktionalanalysis, Approximationstheorie, Numerische Mathematik." (L. Collatz, G. Meinardus, and H. Unger, ed.) pp. 42–47. Birkhäuser Verlag, Basel, 1967.

6. HARVEY S. PRICE AND RICHARD S. VARGA, Numerical analysis of simplified mathematical models of fluid flow in porous media. *Proceedings of Symposia in Applied Mathematics*, Vol. XX (to appear). American Mathematical Society, Providence.

7. RICHARD S. VARGA, On higher order stable implicit methods for solving parabolic partial differential equations. *J. Math. Phys.* **40** (1961), 220–231.

8. J. L. WALSH, The convergence of sequences of rational functions of best approximation with some free poles. *In* "Approximations of Functions." (H. L. Garabedian, ed.) pp. 1–16. Elsevier Publishing Co., New York, 1965.

9. HELMUT WERNER, "Vorlesung über Approximationstheorie." Springer-Verlag, New York, 1966.